

# Separation of Potentials in the Two-Body Problem

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## Abstract

In contrast to the well-known solution of the two-body problem through the use of the concept of reduced mass, a solution is proposed involving separation of potentials. It is shown that each of the two point bodies moves in its own stationary potential well generated by the other body, and the magnitudes of these potentials are calculated. It is shown also that for each body separately the energy and the angular momentum laws are valid. The knowledge of the potentials in which the bodies are moving permits calculation of the trajectories of each body without resorting to the reduced mass.

**Key words:** mechanics, two-body problem, gravitational potential, virial theorem.

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## 1 Introduction

Consider the problem of motion of two point particles with masses  $m_1$ ,  $m_2$  in the absence of external forces (the two-body problem). Here and in what follows, all parameters belonging to the first particle will be labeled by index "1" and those belonging to the second particle, by "2". In order to avoid unnecessary complication of our consideration, we assume the center of mass of the system to be at rest. If the motion of the system is uniform, one

can readily come to the corresponding conclusions by introducing a moving coordinate frame.

We shall look for the solution to the problem in the approximation used, for instance, in monograph [1], i.e., in the nonrelativistic approximation.

There is a well known solution to this problem involving the reduced mass (cf., e.g., [1], §13 or [2], §12). In this case, analysis of the motion of two particles is replaced by consideration of the motion of one fictitious  $\mu$ -particle with a mass  $\mu = \frac{m_1 m_2}{m_1 + m_2}$  (reduced mass), with this  $\mu$ -particle assumed to move in a centrally symmetric field with a center at rest placed at the center of mass of the two particles. The magnitude of this field is governed by the force of interaction of the two particles with one another. On finding the law by which the  $\mu$ -particle moves, one will be able to readily reconstruct the motion of the real particles  $m_1$  and  $m_2$ .

One can offer the other kind of solution. This solution assumes each particle moves in its own stationary potential well and for each body separately the energy and the angular momentum laws are valid. Said otherwise, in place of an analysis of the motion of one  $\mu$ -particle in a gravitational potential field associated with particle interaction one may consider the motion of each real particle in the stationary gravitational potential field created by the other particle. This stationary field differs, however from the gravitational field produced by the  $m_1$  and  $m_2$  masses. Our task is to find this potential fields for every particle.

## 2 Finding the Gravitational Potentials for the Two-Body Problem

Potential energy of interaction of two point masses at rest which are on the distance  $l = |\mathbf{l}|$  from one another is

$$\mathcal{E}_{Pot}(|\mathbf{l}|) = -G \frac{m_1 m_2}{|\mathbf{l}|}, \quad (1)$$

where  $G$  is the gravitational constant, and  $\mathbf{l}$  is the vector from particle "1" to particle "2". This expression one can write in another form

$$\mathcal{E}_{Pot}(|\mathbf{l}|) = m_1 \Phi_2^G(|\mathbf{l}|) = m_2 \Phi_1^G(|\mathbf{l}|). \quad (2)$$

Here  $\Phi_1^G(|\mathbf{l}|)$  and  $\Phi_2^G(|\mathbf{l}|)$  are gravitational potentials created by particles  $m_1$  and  $m_2$  respectively

$$\Phi_1^G(|\mathbf{l}|) = -G\frac{m_1}{|\mathbf{l}|}, \quad \Phi_2^G(|\mathbf{l}|) = -G\frac{m_2}{|\mathbf{l}|}. \quad (3)$$

Coordinate  $|\mathbf{l}|$  is the distance between the particles. It is not convenient to use this coordinate when particles are moving. It is more convenient to use coordinate  $\mathbf{r}$  which one can reckon from the center of mass. Taking it into account we write energies of the particles "1" and "2", which are located at points  $\mathbf{r}_1$  and  $\mathbf{r}_2$  similar to formula (2)

$$\mathcal{E}_{1,Pot}(\mathbf{r}_1) = m_1\Phi_2(\mathbf{r}_1) = m_1K_2\Phi_2^G(\mathbf{r}_1), \quad (4)$$

$$\mathcal{E}_{2,Pot}(\mathbf{r}_2) = m_2\Phi_1(\mathbf{r}_2) = m_2K_1\Phi_1^G(\mathbf{r}_2). \quad (5)$$

Here  $\Phi_1(\mathbf{r})$  and  $\Phi_2(\mathbf{r})$  are the potentials which we must find,  $K_1$  and  $K_2$  are the coefficients connected this potentials with the gravitational potentials generated by the particles  $m_1$  and  $m_2$ :

$$\Phi_1(\mathbf{r}) = K_1 \left( -G\frac{m_1}{r} \right), \quad \Phi_2(\mathbf{r}) = K_2 \left( -G\frac{m_2}{r} \right).$$

This formulation suggests that we are describing not the energy of interaction of particles with one another but rather the potential energy of each particle separately, which is referenced to a certain level. As is common practice, we take for the zero level the energy of interaction of a particle  $m$  with other particle at infinity.

Let us turn now to the above potentials. We shall use for this purpose the relations derived for the well known solution involving the reduced mass.

We place the origin of coordinates at the center of mass of the system under consideration. In this case

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0. \quad (6)$$

Introduce a vector defining the relative positions of the particles

$$\mathbf{l} = \mathbf{r}_2 - \mathbf{r}_1. \quad (7)$$

We have defined this vector by  $\mathbf{l}$  rather than by  $\mathbf{r}$  as this is done usually, in order not to confuse it with the running coordinate  $\mathbf{r}$  which we reckon from the origin. Vector  $\mathbf{l}$  is the same vector which enter in the (1), (2) and (3) formulas.

The equalities (6) and (7) yield

$$\mathbf{r}_1 = -\frac{m_2}{m_1 + m_2}\mathbf{l}, \quad \mathbf{r}_2 = \frac{m_1}{m_1 + m_2}\mathbf{l}. \quad (8)$$

Differentiating Eqs. (6) – (8) with respect to time, we come to similar relations for the point velocities

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = 0, \quad (9)$$

$$\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1, \quad (10)$$

$$\mathbf{v}_1 = -\frac{m_2}{m_1 + m_2}\mathbf{v}, \quad \mathbf{v}_2 = \frac{m_1}{m_1 + m_2}\mathbf{v}. \quad (11)$$

Here  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are the velocities of the particles under consideration, and  $\mathbf{v}$  is the relative particle velocity. Recall that it is the relative quantities  $\mathbf{l}$  and  $\mathbf{v}$  that are invoked in dealing with the problem with the use of reduced mass. And it is the coordinate  $\mathbf{l}$  that is employed to describe the trajectory of the  $\mu$ -particle.

We can use now Eq. (11) to express the kinetic energy of each particle in terms of the relative velocity:

$$\mathcal{E}_{1,Kin}(\mathbf{r}_1) = \frac{m_1}{2}v_1^2(\mathbf{r}_1) = \frac{m_1}{2} \frac{m_2^2}{(m_1 + m_2)^2}v^2, \quad (12)$$

$$\mathcal{E}_{2,Kin}(\mathbf{r}_2) = \frac{m_2}{2}v_2^2(\mathbf{r}_2) = \frac{m_2}{2} \frac{m_1^2}{(m_1 + m_2)^2}v^2. \quad (13)$$

In Eqs. (12) and (13), the kinetic energy is expressed both through the velocities of individual particles and the relative velocity. An analysis of Eqs. (12) and (13) suggests that the expressions for the energy written in terms of the velocities of individual particles and through the relative velocity differ in characteristic factors:

$$\frac{m_1^2}{(m_1 + m_2)^2}, \quad \frac{m_2^2}{(m_1 + m_2)^2}. \quad (14)$$

We are going now to express the potential energy of each particle through its coordinates (see Eqs. (4) and (5)), whereas up to now the potential energy

of interaction of two particles was defined in terms of a relative parameter, namely, separation distance between the particles  $|\mathbf{l}|$  (see Eq. (1)).

This suggests that a transition from description of the energy through separation distance between the particles, a relative parameter, to that in terms of the energy of each particle separately should bring about the appearance of factors of the kind of Eq. (14).

Furthermore, by the virial theorem (see, e.g., [1], §10, or [2], §6), for the Coulomb potential well the kinetic energy is related to the potential energy through

$$\bar{\mathcal{E}}_{Kin} = -\frac{1}{2}\bar{\mathcal{E}}_{Pot},$$

where the line above denotes averaging over time. For circular motion, this relation is correct without averaging. Thus, the potential energy is proportional to the kinetic energy.

This all adds up to the following recasting of Eqs. (4) and (5):

$$\mathcal{E}_{1,Pot}(\mathbf{r}_1) = m_1\Phi_2(\mathbf{r}_1) = m_1\frac{m_2^2}{(m_1+m_2)^2}\left(-G\frac{m_2}{|\mathbf{r}_1|}\right), \quad (15)$$

$$\mathcal{E}_{2,Pot}(\mathbf{r}_2) = m_2\Phi_1(\mathbf{r}_2) = m_2\frac{m_1^2}{(m_1+m_2)^2}\left(-G\frac{m_1}{|\mathbf{r}_2|}\right). \quad (16)$$

We obtain

$$K_1 = \frac{m_1^2}{(m_1+m_2)^2}, \quad K_2 = \frac{m_2^2}{(m_1+m_2)^2}, \quad (17)$$

and potentials

$$\Phi_1(\mathbf{r}) = K_1\Phi_1^G(\mathbf{r}) = \frac{m_1^2}{(m_1+m_2)^2}\left(-G\frac{m_1}{|\mathbf{r}|}\right), \quad (18)$$

$$\Phi_2(\mathbf{r}) = K_2\Phi_2^G(\mathbf{r}) = \frac{m_2^2}{(m_1+m_2)^2}\left(-G\frac{m_2}{|\mathbf{r}|}\right). \quad (19)$$

Both potentials  $\Phi_1(\mathbf{r})$  and  $\Phi_2(\mathbf{r})$  form a stationary potential wells placed at the origin in which the real particles  $m_1$  and  $m_2$  move. Each particle, however, moves in its “own” potential well, i.e., the particle  $m_1$  moves in the potential well formed by the potential  $\Phi_2(\mathbf{r})$ , and the particle  $m_2$ , in the potential well created by the potential  $\Phi_1(\mathbf{r})$ .

The potentials  $\Phi_1(\mathbf{r})$  and  $\Phi_2(\mathbf{r})$  are naturally nothing more than conventional potentials. They have been introduced by convention to describe the motion of the real particles  $m_1$  and  $m_2$ .

### 3 Conservation Laws for Every Particle

The time has come now to check whether the expressions for the potentials  $\Phi_1(\mathbf{r})$  and  $\Phi_2(\mathbf{r})$  are correct. How can one verify the correctness of these relations? First, the sum of the potential energies of both particles should be equal to the potential energy (1) of interaction of the both particles with one another.

Summing Eqs. (15) and (16) in conjunction with Eq. (8), we see clearly that this equality is upheld for all  $\mathbf{r}_1$  and  $\mathbf{r}_2$  at any moment of time.

Because each particle moves in its fixed potential well, the laws of conservation of energy and momentum should be met for each particle separately. Let us check it.

We use Eqs. (12) and (13) to find the ratio of the kinetic energies of the two particles

$$\frac{\mathcal{E}_{1,Kin}(\mathbf{r}_1)}{\mathcal{E}_{2,Kin}(\mathbf{r}_2)} = \frac{m_2}{m_1}, \quad (20)$$

whence

$$\mathcal{E}_{1,Kin}(\mathbf{r}_1) = \frac{m_2}{m_1} \mathcal{E}_{2,Kin}(\mathbf{r}_2), \quad \mathcal{E}_{2,Kin}(\mathbf{r}_2) = \frac{m_1}{m_2} \mathcal{E}_{1,Kin}(\mathbf{r}_1). \quad (21)$$

Next we take Eqs. (15), (16), and (8) to find the ratio of the potential energies of the two particles

$$\frac{\mathcal{E}_{1,Pot}(\mathbf{r}_1)}{\mathcal{E}_{2,Pot}(\mathbf{r}_2)} = \frac{m_2}{m_1}, \quad (22)$$

whence

$$\mathcal{E}_{1,Pot}(\mathbf{r}_1) = \frac{m_2}{m_1} \mathcal{E}_{2,Pot}(\mathbf{r}_2), \quad \mathcal{E}_{2,Pot}(\mathbf{r}_2) = \frac{m_1}{m_2} \mathcal{E}_{1,Pot}(\mathbf{r}_1). \quad (23)$$

Summing Eqs. (21) and (23) term by term, we come to the total energies  $\mathcal{E}_1(\mathbf{r}_1)$  and  $\mathcal{E}_2(\mathbf{r}_2)$  of each particle

$$\mathcal{E}_1(\mathbf{r}_1) = \mathcal{E}_{1,Pot}(\mathbf{r}_1) + \mathcal{E}_{1,Kin}(\mathbf{r}_1) = \frac{m_2}{m_1}(\mathcal{E}_{2,Pot}(\mathbf{r}_2) + \mathcal{E}_{2,Kin}(\mathbf{r}_2)), \quad (24)$$

or, in a more concise form

$$\mathcal{E}_1(\mathbf{r}_1) = \frac{m_2}{m_1}\mathcal{E}_2(\mathbf{r}_2), \quad \mathcal{E}_2(\mathbf{r}_2) = \frac{m_1}{m_2}\mathcal{E}_1(\mathbf{r}_1). \quad (25)$$

Thus, we see that the ratio of the total energies of each particle obeys the equality similar to the relations (20) and (22)

$$\frac{\mathcal{E}_1(\mathbf{r}_1)}{\mathcal{E}_2(\mathbf{r}_2)} = \frac{m_2}{m_1}. \quad (26)$$

The sum of the total energies of each particle should yield the total energy of the whole system. Using equalities (25) we come to

$$\mathcal{E} = \mathcal{E}_1(\mathbf{r}_1) + \mathcal{E}_2(\mathbf{r}_2) = \mathcal{E}_1(\mathbf{r}_1) + \frac{m_1}{m_2}\mathcal{E}_1(\mathbf{r}_1) = \mathcal{E}_1(\mathbf{r}_1)\frac{m_1 + m_2}{m_2}, \quad (27)$$

$$\mathcal{E} = \mathcal{E}_1(\mathbf{r}_1) + \mathcal{E}_2(\mathbf{r}_2) = \mathcal{E}_2(\mathbf{r}_2) + \frac{m_2}{m_1}\mathcal{E}_2(\mathbf{r}_2) = \mathcal{E}_2(\mathbf{r}_2)\frac{m_1 + m_2}{m_1}. \quad (28)$$

The total  $\mathcal{E}$  is conserved as the energy of a closed system. Hence, as follows from equalities (27) and (28), the total energies of each of the particles are conserved, i.e., are coordinate independent

$$\mathcal{E}_1(\mathbf{r}_1) = \mathcal{E}_1 = const., \quad \mathcal{E}_2(\mathbf{r}_2) = \mathcal{E}_2 = const. \quad (29)$$

Thus, each particle moves in its potential well with its own energy.

Consider now the angular momentum of the system and of the particles separately. We express the angular momentum of particles through the coordinates  $\mathbf{l}$  and the relative velocity  $\mathbf{v}$ . Combining Eqs. (8) and (11), we write the angular momenta of the particles as follows

$$\mathbf{M}_1(\mathbf{r}_1) = m_1[\mathbf{r}_1 \times \mathbf{v}_1] = m_1 \frac{m_2^2}{(m_1 + m_2)^2} [\mathbf{l} \times \mathbf{v}], \quad (30)$$

$$\mathbf{M}_2(\mathbf{r}_2) = m_2[\mathbf{r}_2 \times \mathbf{v}_2] = m_2 \frac{m_1^2}{(m_1 + m_2)^2} [\mathbf{l} \times \mathbf{v}]. \quad (31)$$

As seen from Eqs. (30) and (31), the vectors  $[\mathbf{r}_1 \times \mathbf{v}_1]$ ,  $[\mathbf{r}_2 \times \mathbf{v}_2]$  and  $[\mathbf{l} \times \mathbf{v}]$  are oriented in the same direction. From the ratio of the momenta

$$\frac{\mathbf{M}_1(\mathbf{r}_1)}{\mathbf{M}_2(\mathbf{r}_2)} = \frac{m_2}{m_1}, \quad (32)$$

we immediately obtain

$$\mathbf{M}_1(\mathbf{r}_1) = \frac{m_2}{m_1} \mathbf{M}_2(\mathbf{r}_2), \quad \mathbf{M}_2(\mathbf{r}_2) = \frac{m_1}{m_2} \mathbf{M}_1(\mathbf{r}_1). \quad (33)$$

The sum of the momenta of the two particles should give the total angular momentum of the system. We sum the momenta to obtain

$$\mathbf{M} = \mathbf{M}_1(\mathbf{r}_1) + \mathbf{M}_2(\mathbf{r}_2) = \mathbf{M}_1(\mathbf{r}_1) + \frac{m_1}{m_2} \mathbf{M}_1(\mathbf{r}_1) = \mathbf{M}_1(\mathbf{r}_1) \frac{m_1 + m_2}{m_2}, \quad (34)$$

$$\mathbf{M} = \mathbf{M}_1(\mathbf{r}_1) + \mathbf{M}_2(\mathbf{r}_2) = \mathbf{M}_2(\mathbf{r}_2) + \frac{m_2}{m_1} \mathbf{M}_2(\mathbf{r}_2) = \mathbf{M}_2(\mathbf{r}_2) \frac{m_1 + m_2}{m_1}. \quad (35)$$

The total angular momentum  $\mathbf{M}$ , as the momentum of a closed system, is conserved. But then, as seen from the equalities (34) and (35), the momenta of each of the particles are conserved, i.e., they are coordinate independent:

$$\mathbf{M}_1(\mathbf{r}_1) = \mathbf{M}_1 = \text{const.}, \quad \mathbf{M}_2(\mathbf{r}_2) = \mathbf{M}_2 = \text{const.} \quad (36)$$

Because for each particle both the total energy and the angular momentum are conserved, one may consider the motion of each particle in its potential well separately, and calculate the energies and angular momenta also separately for each particle. Also, there is no need to use the reduced mass; indeed, one may use instead real masses of each particle. Relation of energies of the particles must obey Eqs. (26), (27) and (28), and relation of angular momenta – Eqs. (32), (34) and (35).

If, however, one has to know the positions of particles with respect to one another, one will have to consult Eqs. (6) and (9).

We note in conclusion that this approach permits one to separate completely not only the kinetic but the potential energies as well. Indeed, instead of expressing the potential energy through the relative coordinate  $\mathbf{l}$  (see Eq. (1)), it can be defined in terms of the coordinates of each particle separately

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = \mathcal{E}_{1,Kin}(\mathbf{v}_1) + \mathcal{E}_{1,Pot}(\mathbf{r}_1) + \mathcal{E}_{2,Kin}(\mathbf{v}_2) + \mathcal{E}_{2,Pot}(\mathbf{r}_2).$$



## 4 Trajectories of Particle Motion

Because in each potential well particles move saving its energies and angular momenta one can study the behavior of particles with the use of the well known relations describing the motion of one particle in a potential well (see, e.g., [1], §15). As follows from these relations, if the potential energy in a central field is inversely proportional to  $r$  ( $r$  is the radial coordinate), i.e., it can be written as (showing explicitly the sign)

$$\mathcal{E}_{Pot}(r) = -\frac{\alpha}{r}, \quad (37)$$

where  $\alpha$  is a positive constant, then the particle trajectory in polar coordinates  $r, \varphi$  has the shape of an ellipse with the focus at the origin

$$r = \frac{p}{1 + e \cos(\varphi - c)}. \quad (38)$$

Here  $c$  is the constant of integration, and  $p$  and  $e$  are the orbital parameter and the eccentricity

$$p = \frac{M^2}{m\alpha}, \quad (39)$$

$$e = \sqrt{1 + \frac{2\mathcal{E}M^2}{m\alpha^2}}. \quad (40)$$

Here  $\mathcal{E}$  is the total energy of the particle, and  $M^2$  is the squared angular momentum. If it is necessary to find the trajectory of the particle one must know quantities  $\mathcal{E}$  and  $M^2$ .

It is known that the same relations can be used if one analyzes the relative motion of two particles. In this case, one has to put for mass  $m$  in these relations the reduced mass  $\mu = \frac{m_1 m_2}{m_1 + m_2}$ . A fictitious particle with mass  $\mu$  describes the motion of a system of two particles. We are going in what follows to label all quantities related to the fictitious particle  $\mu$  by index  $\mu$ .

Let us see how the trajectories of each particle are related to that of the fictitious particle  $\mu$  describing the behavior of a system of two particles. And also, whether it is possible to trace the possible trajectories of each particle without resorting to the intermediate  $\mu$  particle.

If the axis from which the angle  $\varphi$  is reckoned passes through the point of the ellipse closest to the focus (the perihelion), the constant of integration  $c$

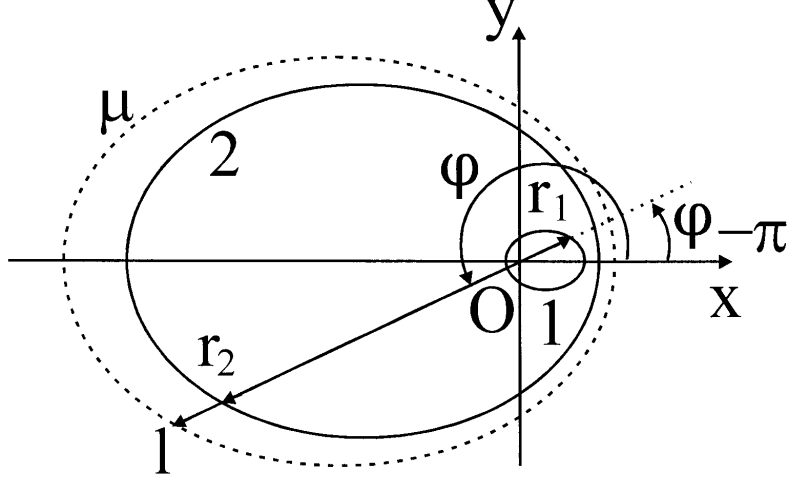


Figure 1:

in Eq. (38) is zero, and if this point is farthest from the focus of the ellipse,  $c = \pm\pi$  (with either of the signs). The perihelia of the orbits of particles “1” and “2” look, however, in opposite directions. This is seen clearly already from equality (6); indeed, for this equality to hold, vectors  $\mathbf{r}_1$  and  $\mathbf{r}_2$  must reach the maximum and the minimum simultaneously. The orbits of the particles are shown in Fig. 1.

In Fig. 1, the center of mass of the system is at point  $O$  (the origin). Also shown are the orbits of particles “1” and “2”, vector  $\mathbf{l} = \mathbf{r}_2 - \mathbf{r}_1$  and the  $\mu$  orbit (dashed) along which the fictitious particle with mass  $\mu$  moves. As seen from Fig. 1, the constant  $c$  for the ellipse “1” should be equal to  $c_1 = \pm\pi$ , and for the ellipse “2”,  $c_2 = 0$ .

We are going to demonstrate that the above approach offers a possibility of calculating possible trajectories of each particle without resorting to the reduced mass.

Apply now Eqs. (37)–(40) to a study of the motion of our particles “1” and “2”. As seen from Eqs. (15) and (16), the parameter  $\alpha$  for our particles can be written

$$\alpha_1 = \frac{Gm_1m_2m_2^2}{(m_1 + m_2)^2}, \quad \alpha_2 = \frac{Gm_1m_2m_1^2}{(m_1 + m_2)^2}, \quad (41)$$

whence

$$\frac{\alpha_1}{\alpha_2} = \frac{m_2^2}{m_1^2}, \quad \alpha_1 = \frac{m_2^2}{m_1^2} \alpha_2, \quad \alpha_2 = \frac{m_1^2}{m_2^2} \alpha_1. \quad (42)$$

Using Eqs. (26), (32), and (42), one can readily verify that  $e_1 = e_2$ , i.e., the eccentricities for both particles are equal. This implies that the shape of the trajectory (shape of the ellipse) is the same for the two particles. As for the  $p$  parameters, i.e., the dimensions of the ellipses, these parameters obey the relation

$$\frac{p_1}{p_2} = \frac{m_2}{m_1}, \quad (43)$$

whence

$$p_1 = \frac{m_2}{m_1} p_2, \quad p_2 = \frac{m_1}{m_2} p_1 \quad (44)$$

Accordingly, all the parameters related to the ellipse dimensions (the major and minor semiaxes and others) obey the same relation.

Express vector  $\mathbf{l}$  from Eq. (7) through the ellipse parameters (the coordinate  $\mathbf{l}$  describes the trajectory of the  $\mu$  particle). We can use the general relation (38) to express  $\mathbf{r}_1$  and  $\mathbf{r}_2$  through the parameters of the ellipses. Because vectors  $\mathbf{r}_1$  and  $\mathbf{r}_2$  are antiparallel, in polar coordinates the vectors will differ by  $\pi$ . For the coordinate  $r$  of the trajectory of the  $\mu$  particle we have  $r = |\mathbf{l}| = |\mathbf{r}_2 - \mathbf{r}_1| = |\mathbf{r}_2| + |\mathbf{r}_1|$ . The angular coordinate of vector  $\mathbf{r}_2$  is  $\varphi_2 = \varphi$ , because it is for the “2” ellipse that the constant  $c_2 = 0$ . For the vector  $\mathbf{r}_1$ , the angular coordinate  $\varphi_1 = \varphi - \pi$ . Apart from this, as already mentioned, for the ellipse “1” the constant  $c_1 = \pm\pi$ . Using Eqs. (44) and recalling that the directions of the vectors differ by  $\pi$ , we come to

$$r = r_\mu = \frac{p_2}{1 + e \cos \varphi} + \frac{p_1}{1 + e \cos((\varphi - \pi) \pm \pi)} = \frac{m_1}{m_2} \frac{p_1}{(1 + e \cos \varphi)} + \frac{p_1}{1 + e \cos \varphi},$$

or

$$r_\mu = \frac{m_1 + m_2}{m_2} \frac{p_1}{(1 + e \cos \varphi)}, \quad r_\mu = \frac{m_1 + m_2}{m_1} \frac{p_2}{(1 + e \cos \varphi)}. \quad (45)$$

Thus, the trajectory of the  $\mu$  particle is also an ellipse, with the same eccentricity as those of particles “1” and “2”, and with an orbit parameter (denote it by  $p_\mu$ )

$$p_\mu = \frac{m_1 + m_2}{m_2} p_1, \quad p_\mu = \frac{m_1 + m_2}{m_1} p_2. \quad (46)$$

Express the parameters of the ellipse  $\mu$  in terms of those of ellipses “1” and “2”. The angular momentum of the system should be the sum of momenta of the particles and equal to the momentum of the  $\mu$ -particle, because it is the latter that describes the behavior of the system:

$$\mathbf{M} = \mathbf{M}_\mu = \mathbf{M}_1 + \mathbf{M}_2.$$

We have already calculated the angular momentum of the system (Eqs. (34) and (35)). Squaring  $\mathbf{M}_\mu$  and using equality (33), we come to

$$\mathbf{M}_\mu^2 = \mathbf{M}_1^2 + \mathbf{M}_2^2 + 2\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_1^2 + \mathbf{M}_2^2 + 2\frac{m_1}{m_2}\mathbf{M}_1^2 = M_1^2 + \frac{m_1^2}{m_2^2}M_1^2 + 2\frac{m_1}{m_2}M_1^2. \quad (47)$$

Recasting expression (39) to the form  $M_1^2 = p_1 m_1 \alpha_1$ , we substitute in expression (47)  $p_1$  from relation (46), and  $\alpha_1$  from relation (41), and take into account that for system of two particles  $\alpha = Gm_1 m_2$ . The end result is

$$\mathbf{M}_\mu^2 = p_\mu \frac{m_1 m_2}{m_1 + m_2} G m_1 m_2 = p_\mu \mu \alpha. \quad (48)$$

Thus we have once again come to the well known result, namely, if we express the parameters of the ellipse along which the  $\mu$ -point moves through the parameters of motion of real particles, then for the mass we should take  $\mu = \frac{m_1 m_2}{m_1 + m_2}$ , i.e., the reduced mass.

Summing up, we have shown that in the “two-body problem” one can operate without using the concept of the reduced mass. In this case, however, one should use not the energy of particle interaction (37) but rather the energy of each particle, which is reckoned from the zero level:

$$\mathcal{E}_{1,Pot}(r) = -\frac{Gm_1 m_2 m_2^2}{(m_1 + m_2)^2} \frac{1}{r}, \quad \mathcal{E}_{2,Pot}(r) = -\frac{Gm_1 m_2 m_1^2}{(m_1 + m_2)^2} \frac{1}{r}. \quad (49)$$

For the zero level one takes here the energy of interaction between particles infinitely far from one another.

Energies  $\mathcal{E}_{1,Pot}(r_1)$  and  $\mathcal{E}_{2,Pot}(r_2)$  are not independent. We use formula (6) of the relation of particles coordinates  $\mathbf{r}_1$  and  $\mathbf{r}_2$  to find relation of potential energies of both particles

$$\frac{\mathcal{E}_{1,Pot}(r_1)}{\mathcal{E}_{2,Pot}(r_2)} = \frac{m_2}{m_1}. \quad (50)$$

Naturally, this expression coincide with the equation (22). If it is necessary to know not the energy only but the shape of the trajectories one must take into account relation (32).

It might seem at first glance that this complicates solution of the problem. This approach permits one, however, to separate the potential energies and consider separately the motion of each particle in its potential well.

We note in conclusion that using the standard approach with the reduced mass requires a two-step procedure, in which one first finds the solution for the reduced mass, and after that, derive from this solution the parameters of motion of the particles of interest. Application of the above technique permits one to calculate *directly* the possible trajectories of *each* particle.

Such approach one can use in the case when particles have electric charges  $q_1$  and  $q_2$  (in the case under consideration charges must have opposite signs). Such approach can be used only if velocities of the charges  $v \ll c$  ( $c$  is the light velocity). In this case we can neglect by the energy of the magnetic field which appears in moving charged particles in comparison with the energy of the electric field. Said otherwise, in this case we neglect by the vector potential  $\mathbf{A}(\mathbf{r})$  and claim that particles move in scalar potential  $\Phi(\mathbf{r})$  only.

As a rule in this case one can neglect by the gravitational interaction in comparison with the electromagnetic one. Then instead the gravitational potential

$$\Phi^G(\mathbf{r}) = \left( -G \frac{m}{|\mathbf{r}|} \right)$$

it is necessary to put electric field potential in all formulas (we use the Gaussian absolute system of units)

$$\Phi(\mathbf{r}) = \frac{q}{|\mathbf{r}|}.$$

Then quantity  $\alpha$  from the Eq. (37) is  $\alpha = |q_1||q_2|$ , quantities  $\alpha_1$  and  $\alpha_2$  from the Eq. (41) are:

$$\alpha_1 = \frac{|q_1||q_2|m_2^2}{(m_1 + m_2)^2}, \quad \alpha_2 = \frac{|q_1||q_2|m_1^2}{(m_1 + m_2)^2}. \quad (51)$$

Energies of the particles can be written as (see Eq. (49))

$$\mathcal{E}_{1,Pot}(r_1) = -\frac{|q_1||q_2|m_2^2}{(m_1 + m_2)^2} \frac{1}{r_1}, \quad \mathcal{E}_{2,Pot}(r_2) = -\frac{|q_1||q_2|m_1^2}{(m_1 + m_2)^2} \frac{1}{r_2}. \quad (52)$$

We use Eq. (6) to find ratio of the potential energies of the two particles in this case

$$\frac{\mathcal{E}_{1,Pot}(r_1)}{\mathcal{E}_{2,Pot}(r_2)} = \frac{m_2}{m_1}. \quad (53)$$

Hence in this case ratio of the potential energies of the two revolving particles are inversely proportional to their mass also.

As was shown above coordinate  $r$  of the trajectory of the  $\mu$ -particle is  $r = r_\mu = |l|$ . We use Eq. (8) to sum the potential energies of every particle:

$$\mathcal{E}_{1,Pot}(r_1) + \mathcal{E}_{2,Pot}(r_2) = -\frac{|q_1||q_2|}{l} = -\frac{|q_1||q_2|}{r_\mu}. \quad (54)$$

We receive Eq. (37) that is the energy of interaction of the two particles.

## References

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